CATEGORY THEORY
TOPIC 31 - CATEGORIES

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1. Definition of Category

Category theory is a way to classify and compare abstract objects that naturally occur in mathematics. Each category specified what are the allowed morphisms, which one may think of as arrows between the objects. We will focus on concrete categories, where the objects are sets and the morphisms are functions. We begin by stating the formal definition of category.

Definition 1. A *category* $\mathfrak{C}$ consists of:

- a class of *objects* $\text{Obj}_{\mathfrak{C}}$;
- for any pair of objects $A, B \in \text{Obj}_{\mathfrak{C}}$, a set of *morphisms* $\text{Mor}_{\mathfrak{C}}(A, B)$ from A to B, such that $\text{Mor}_{\mathfrak{C}}(A, B) \cap \text{Mor}_{\mathfrak{C}}(C, D) \neq \varnothing \Rightarrow A = C$ and $B = D$;
- for any three objects $A, B, C \in \text{Obj}_{\mathfrak{C}}$, a law of composition
  $$\circ : \text{Mor}_{\mathfrak{C}}(B, C) \times \text{Mor}_{\mathfrak{C}}(A, B) \to \text{Mor}_{\mathfrak{C}}(A, C)$$
  which is associative, that is, if $f \in \text{Mor}_{\mathfrak{C}}(A, B)$, $g \in \text{Mor}_{\mathfrak{C}}(B, C)$, and $h \in \text{Mor}_{\mathfrak{C}}(C, D)$, then $(h \circ g) \circ f = h \circ (g \circ f)$;
- for each $A \in \text{Obj}_{\mathfrak{C}}$, a morphism $\text{id}_A \in \text{Mor}_{\mathfrak{C}}(A, A)$ satisfying
  (a) if $f \in \text{Mor}_{\mathfrak{C}}(B, A)$, then $\text{id}_A \circ f = f$;
  (b) if $g \in \text{Mor}_{\mathfrak{C}}(A, B)$, then $g \circ \text{id}_A = g$.

A category is called *concrete* if the objects are sets and the morphisms are functions between the sets.

To identify a concrete category, one first identifies the objects. These will be sets with some sort of additional structure; the type of structure is what distinguishes the category. For example, a partial order, or a binary operation, would be considered additional structure. Then one identifies which functions between the objects will be said to "preserve the structure". There are choices to be made here; often there is more than one valuable choice, in which case, one may define more than one category with the same class of objects, but with differing morphisms.

One should be aware that for concrete categories, the third axiom of the definition is automatically satisfied, since function composition always exists and is always associative. The fourth axiom requires that the identity map on the underlying set of an object is considered to be a morphism.

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2. Common Categories

We list some common concrete categories.

Objects	Morphisms
Sets	Functions
Posets	Order preserving maps
Equisets	Partition preserving maps
Graphs	Edge preserving maps
Groups	Group homomorphisms
Abelian Groups	Group homomorphisms
Rings	Ring homomorphisms
Fields	Ring homomorphisms
Vector Spaces	Linear Transformations
Metric Spaces	Continuous functions
Metric Spaces	Isometries
Topological Spaces	Continuous functions
Measure Spaces	Measurable Functions
Probability Spaces	Measurable Functions

3. Subcategories

Definition 2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. We say that $\mathfrak{D}$ is a *subcategory* of $\mathfrak{C}$ if

- $\mathrm{Obj}_{\mathfrak{D}} \subset \mathrm{Obj}_{\mathfrak{C}}$;
- $\mathrm{Mor}_{\mathfrak{D}}(A, B) \subset \mathrm{Mor}_{\mathfrak{C}}(A, B)$ for every $A, B \in \mathrm{Obj}_{\mathfrak{D}}$;
- the laws of composition in $\mathfrak{D}$ are the same as those of $\mathfrak{C}$ whenever applicable.

A subcategory is *wide* if it contains all of the parent category's objects.

A subcategory is *full* if it contains all of the parent category's morphisms, for the objects that are in the subcategory.

A subcategory is given by specifying its objects and its morphisms. However, a full subcategory is determined by the objects in it, and a wide subcategory is determined by the morphisms in it.

We have listed multiple examples of subcategories. The category of abelian groups is a full subcategory of the category of groups. The category of fields is a full subcategory of the category of rings. The category of metric spaces with continuous functions is a full subcategory of the category of topological spaces. The category of metric spaces with isometries is a wide subcategory of the category of metric spaces with continuous functions.

4. Functors

Just as set theory makes the formal study of functions possible, so category theory makes the formal study of functors possible. A functor is like of morphism between categories. Functors play a crucial role is algebraic topology and algebraic geometry, and are useful is areas of mathematics from analysis to probability to computer science. For the sake of completeness and reference, we give the definition of functor here, but we will only refer to this in passing for a while.

Definition 3. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories.

A *covariant functor* F from $\mathfrak{C}$ to $\mathfrak{D}$ is an assignment of every object in $\mathfrak{C}$ to an object in $\mathfrak{D}$, and an assignment of every morphism $f : A \to B$, where $A, B \in \text{Obj}(\mathfrak{C})$, to a morphism $F(f) : F(A) \to F(B)$, such that

(1) $f(\text{id}_A) = \text{id}_{F(A)}$;
(2) $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* G from $\mathfrak{C}$ to $\mathfrak{D}$ is an assignment of every object in $\mathfrak{C}$ to an object in $\mathfrak{D}$, and an assignment of every morphism $f : A \to B$, where $A, B \in \text{Obj}(\mathfrak{C})$, to a morphism $G(f) : G(B) \to G(A)$, such that

(1) $f(\text{id}_A) = \text{id}_{G(A)}$;
(2) $G(g \circ f) = G(f) \circ G(g)$.

5. Isomorphisms, Endomorphisms, and Automorphisms

Category theory allows us to come up with a consistent collection of jargon which may be used in multiple contexts. The first example of this relates to classifying morphisms.

Definition 4. Let $\mathfrak{C}$ be a category and let $A, B \in \text{Obj}_{\mathfrak{C}}$.
The notation $f : A \to B$ means that $f \in \text{Mor}(A, B)$.

A morphism $f : A \to B$ is an *isomorphism* if there exists a morphism $g : B \to A$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$. In this case, we say that f is *invertible* and write f^{-1} for g. The set of isomorphisms from A to B is denoted $\text{Iso}(A, B)$.

An *endomorphism* is a morphism from an object to itself. The set of endomorphisms of A is denoted $\text{End}(A)$.

An *automorphism* is an isomorphism from an object to itself. The set of automorphisms of A is denoted $\text{Aut}(A)$.

Let $\mathfrak{C}$ be a category and let $A \in \text{Obj}_{\mathfrak{C}}$. Then $\text{End}(A)$ is a monoid under composition; the set of invertible elements of $\text{End}(A)$ is $\text{Aut}(A)$, which is a group under composition.

Proposition 1. *Let $\mathfrak{C}$ be a category and let $A, B \in \text{Obj}_{\mathfrak{C}}$. Suppose that $f : A \to B$ is an isomorphism. Then*

$$\text{Iso}(A, B) = \{g \circ f \in \text{Mor}(A, B) \mid g \in \text{Aut}(B)\}.$$

Proof. Call the set on the right hand side Z.

Let $h \in \text{Iso}(A, B)$. Then $h \circ f^{-1}$ is an automorphism of B, with inverse $f \circ h^{-1}$. Let $g = h \circ f^{-1}$. Then $g \circ f = h$, so $h \in Z$.

Let $h \in Z$. Then $h = g \circ f$ for some $g \in \text{Aut}(B)$. Then h is an isomorphism, with inverse $f^{-1} \circ g^{-1}$. $\square$

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